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EQUATION FOR STOCHASTIC THERMAL FIELDS

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From phenomenological laws in the difference representation for the mean values and on the assumption that the thermal field is Markovian we obtained a partial differential equation with account for the field stochasticity. Integral mean values of the thermal field are shown to obey the well-known heat conduction equation.

Introduction. As is well known [1], stochastic fields ξ that in the space $\bar{q} = (q_1, q_2, q_3) \in D \subset R^3$ and time $t \in [0, \infty]$ assume T values can be described theoretically via a probability density functional $P(T)$ that obeys the Kolmogorov–Fokker–Plank equation in partial functional derivatives. Because of certain difficulties that were discussed in [1] this equation is practically not used for calculations in solving applied problems.

Apart from the Kolmogorov method, stochasticity can be taken into account 1) with the aid of Ito or Stratonovich stochastic integrals [2], 2) by deriving an equation for a nonequilibrium distribution function from the Liouville equation [3], and 3) with the aid of equations for probability waves [4-9].

In addition to these methods, for describing stochastic fields an approach seems promising that is based on simultaneous analysis of local relations for the mean values of the thermal field that conform to the Fourier law and the continuity equation, and of expressions for the mean temperatures obtained from an explicit difference scheme of the Kolmogorov–Fokker–Planck equation for a stochastic process at each space point. The present paper aims at implementing this approach in constructing the following problem: to find the probability density function for a stochastic thermal field as a solution of a partial differential equation on condition that the values of this function are known at the initial instant of time and at the boundary $\Gamma_\Lambda(\bar{q}) \in R^3$.

1. Probability Density Function for a Scalar Stochastic Field. For the subsequent presentation we agree upon the following notation. We consider the measurable probability space $\{\Omega, A, P\}$ mapped into the measurable space (T, β), where A and β are Boolean algebras, Ω is the space of elementary events $\omega \in \Omega$, P is the probability, $T \in T \subset R^1$ is the set of values of a certain scalar stochastic field $\xi(\omega, \bar{q}, t)$ in the space $\bar{q} = (q_1, q_2, q_3) \in \Lambda(\bar{q}) < R^3$ and time $t \in [0, \omega]$, and $\Lambda(\bar{q}) \in R^3$ is a closed space region with the boundary $\Gamma_\Lambda(\bar{q}) \in R^3$.

Definition 1. The integral probability function for the stochastic field $\xi(\omega, \bar{q}, t)$ is the probability that the stochastic field at any point $\bar{q} \in \Lambda \in R^3$ at any instant of time $t > 0$ assumes a value that is smaller than T:

$$\Phi_\xi(\bar{q}, t, T) = P\left\{\forall t \in [0, \infty)\,,\ \forall \bar{q} \in \Lambda,\ \xi(\omega, \bar{q}, t) < T\right\}. \tag{1}$$

With account for the probabilistic meaning of the function $\Phi_\xi(\bar{q}, t, T)$, it has the following properties:

1) $\Phi_\xi(\bar{q}, t, T)$ – is nondecreasing in the argument T:

$$\Phi_\xi(\bar{q}, t, T_1) \le \Phi_\xi(\bar{q}, t, T_2) \quad \text{for} \quad T_1 < T_2\,,\quad \forall \bar{q} \in \Lambda\,,\quad \forall t \in [0, \infty)\,; \tag{2}$$

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$$2)\ \lim_{T\to\pm\infty} \Phi_\xi(\bar{q}, t, T) = 1\ , \quad \forall\bar{q}\in\Lambda\ , \quad \forall t\in[0,\infty)\ ; \tag{3}$$

$$3)\ \Phi_\xi(\bar{q}, t, T) \in C^2(\bar{q}\in\Lambda, t\in[0,\infty)\ , T)\ . \tag{4}$$

Definition 2. The function $\Pi_\xi(\bar{q},\ t;\ T)$ that is defined by the equation

$$\Pi_\xi(\bar{q}, t; T) = \frac{\partial\Phi_\xi(\bar{q}, t; T)}{\partial T}\ , \tag{5}$$

is called the probability density function for the stochastic field $\xi(\omega,\ \bar{q},\ t)$.

From Definitions 1 and 2 it follows that $\Pi_\xi(\bar{q},\ t;\ T)$ is nonnegative and depends on both the space-time coordinates $(\bar{q}, t)$ and the mappings of T of the stochastic field, with T entering into $\Pi_\xi(\bar{q},\ t;\ T)$ as an independent argument of $\bar{q}$ and t.

Hereafter we will assume that the stochastic fields considered are generalized Markovian fields [2], and the probability density function $\Pi_\xi(\bar{q},\ t;\ T)$ is twice continuously differentiable with respect to $q_i (i = \overline{1,3})$ and T and once continuously differentiable with respect to the time t.

Definition 3. The mean value of the stochastic field $\xi(\omega,\ \bar{q},\ t)$ (or the moment of the first order) is the quantity

$$\langle\xi(\bar{q}, t)\nabla = \int_{-\infty}^{+\infty} T\,\Pi_\xi(\bar{q}, t; T)\,dT\ . \tag{6}$$

2. Derivation of an Equation for the Probability Density Function of a Stochastic Thermal Field. Let us introduce the set of discrete and continuous independent variables $\Theta_\Lambda = \{q_i = ih,\ i = \overline{0,N};\ t_j = j\tau,\ j = \overline{0,J},\ -\infty < T < \infty\}$ and select the pattern $(q_{i+1},\ t_j;\ T)$, $(q_i,\ t_j;\ T)$, $(q_{j-1},\ t_j;\ T)$, and $(q_i,\ t_{j+1};\ T)$.

An analysis of deriving heat conduction equations [10] permits one to draw the following conclusion. The mean temperatures inside each cell of the grid can be calculated as follows. For all nodes on the j-th time layer, except for the node $(q_i,\ t_j;\ T)$, the mean temperature is the result of multiplication of T by the conditional probability, each of which is the product of the probability $\Pi_\xi(q_{i\pm1},\ t_j;\ T)dT$ of the state of the system at a given point q_i and a given time t_j with a given temperature T by the probability $p = a\tau/h^2$ of a change in this state over the time τ by spatial heat transfer from the central node over the distance h by the heat transfer laws. For the central node $(q_i,\ t_j;\ T)$, the mean temperature is obtained by multiplying T by a quantity that is the difference between unity and the conditional probabilities of the outflow from this node into two neighboring ones. Obviously, this difference should be positive. The requirement of positiveness of the above difference coincides with the condition of stability for the explicit difference-differential scheme considered below:

$$T\Pi_\xi(q_i,\ t_j;\ T)dT\{1 - 2a\frac{\tau}{h^2}\}.$$

At each space node, the Markovian field transforms into a Markovian process that conforms to the Kolmogorov–Fokker–Planck equation, which we represent in the difference-differential form

$$\Pi_\xi(q_i, t_{j+1}; T) = \Pi_\xi(q_i, t_j; T) -$$

$$-\tau\frac{\partial}{\partial T}\{f(q_i, t_j; T)\,\Pi_\xi(q_i, t_j; T)\} + \tau\frac{\partial^2}{\partial T^2}\{B(q_i, t_j; T)\,\Pi_\xi(q_i, t_j; T)\}\ .$$

Hence, because the process at the node $(q_i, t_{j+1}; T)$ is Markovian, we obtain the following relation for the local mean temperatures:

$$T\,\Pi_\xi(q_i, t_{j+1}; T)\,dT = T\,\Pi_\xi(q_i, t_j; T)\,dT -$$

$$-\tau T dT \frac{\partial}{\partial T}\left\{f(q_i, t_j; T)\,\Pi_\xi(q_i, t_j; T)\right\} + \tau T dT \frac{\partial^2}{\partial T^2}\left\{B(q_i, t_j; T)\,\Pi_\xi(q_i, t_j; T)\right\}.$$

As a result of simultaneous analysis of phenomenological relations and relations following from the assumption that the thermal field is Markovian, we obtain the following expression for the mean value of the thermal field at the node $(q_i, t_{j+1}; T)$:

$$T\,\Pi_\xi(q_i, t_{j+1}; T)\,dT = T\,\Pi_\xi(q_i, t_j; T)\,dT\left\{1 - 2a\frac{\tau}{h^2}\right\} +$$

$$+T\left\{\frac{a\tau}{h^2}\Pi_\xi(q_{j+1}, t_j; T)\,dT + \frac{a\tau}{h^2}\Pi_\xi(q_{j-1}, t_j; T)\,dT\right\} -$$

$$-T\tau\frac{\partial}{\partial T}\left\{f(q_i, t_j; T)\,\Pi_\xi(q_i, t_j; T)\right\}dT +$$

$$+T\tau\frac{\partial^2}{\partial T^2}\left\{B(q_i, t_j; T)\,\Pi_\xi(q_i, t_j; T)\right\}dT. \tag{7}$$

Suppose we know a priori the initial distributions of the probability density functions of temperature

$$\Pi_\xi(q_i, t_j = t_0 = 0; T) = \Phi(q_i, T),\ i \in [0, I],\ T \in (-\infty, \infty) \tag{8}$$

and the behavior of the probability density function at boundary nodes of the grid

$$\Pi_\xi(q_0, t_j; T) = \Omega(t_j, T),\ \Pi_\xi(q_N, t_j; T) = Z(t_j, T),$$
$$j \in (0, J],\ T \in (-\infty, \infty). \tag{9}$$

On infinity

$$\lim_{T\to\pm\infty} \Pi_\xi(q_i, t_j; T) = 0. \tag{10}$$

Passing to the limits as $h \to 0$ and $\tau \to 0$, it is easy to reduce problem (7)-(10) to the following initial-boundary-value problem for a partial differential equation for the probability density function:

$$\frac{\partial \Pi_\xi(q, t; T)}{\partial t} = a\frac{\partial^2 \Pi_\xi(q, t; T)}{\partial q^2} - \frac{\partial(f(q, t; T)\,\Pi_\xi(q, t; T))}{\partial T} +$$

$$+\frac{\partial^2(B(q, t; T)\,\Pi_\xi(q, t; T))}{\partial T^2}, \tag{11}$$

$$T \in (-\infty, \infty),\ t > 0,\ q \in (q_0, q_N);$$

$$\Pi_\xi (q, t=0 ; T) = \Phi (q, T) ; \; q \in [q_0, q_N] , \; T \in (-\infty, \infty) ; \tag{12}$$

$$\Pi_\xi (q = q_0, t; T) = \Omega (t, T) ; \; t \in [0, t_{\max}] , \; T \in (-\infty, \infty) ; \tag{13}$$

$$\Pi_\xi (q = q_N, t; T) = Z (t, T) ; \; t \in [0, t_{\max}] , \; T \in (-\infty, \infty) ; \tag{14}$$

$$\lim_{T \to \pm\infty} \Pi_\xi (q, t; T) = 0 , \; \forall q \in [q_0, q_N] , \; \forall t \geq 0 . \tag{15}$$

It should be noted that functions that form unambiguity conditions, must have all the properties of a probability density function that have been described in Para. 1 of the present paper, in particular, they must satisfy normalization conditions.

3. Connection of the Equation for the Probability Density Function to the Equation for the Mean Values of the Thermal Field. Without loss of generality, the subsequent discussion will be performed for the case of a single space variable.

Theorem. *If the function $\Pi_\xi(q, t; T)$ satisfies Eqs. (11)-(15) and the conditions*

$$\Pi_\xi (q, t; T) \geq 0 , \; q \in [q_0, q_N] , \; t \in (0, +\infty) , \; T \in (-\infty, +\infty) ; \tag{16}$$

$$\int_{-\infty}^{+\infty} \Pi_\xi (q, t = 0; T) \, dT = 1 , \; \forall q \in [q_0, q_N] ; \tag{17}$$

$$\int_{-\infty}^{+\infty} \Pi_\xi (q, t; T) \big|_{q=q_0, q_N} dT = 1 , \; \forall t > 0 ; \tag{18}$$

$$\lim_{T \to \pm\infty} T f (q, t; T) \, \Pi_\xi (q, t; T) = 0 ; \tag{19}$$

$$\lim_{T \to \pm\infty} T \frac{\partial}{\partial T} (B (q, t; T) \, \Pi_\xi (q, t; T)) = 0 , \tag{20}$$

then $\Pi_\xi(q, t; T)$ satisfies the normalization condition

$$\int_{-\infty}^{+\infty} \Pi_\xi (q, t; T) \, dT = 1 , \; \forall q \in [q_0, q_N] , \; \forall t \in [0, +\infty) \tag{21}$$

and is a probability density function, and the mean value $\langle \xi(q, t) \rangle$ that is determined from Eq. (6) satisfies the initial-boundary-value problem for the heat conduction equation

$$\frac{\partial \langle \xi (q, t) \rangle}{\partial t} - a \frac{\partial^2 \langle \xi (q, t) \rangle}{\partial q^2} = \langle f (q, t, \xi) \rangle , \; t > 0 , \; q \in [q_0, q_N] \tag{22}$$

with the boundary conditions of the first kind

$$\langle \xi (q = q_0, t) \rangle = \int_{-\infty}^{+\infty} T\Omega (t; T)\, dT\,, t > 0\,, \tag{23}$$

$$\langle \xi (q = q_N, t) \rangle = \int_{-\infty}^{+\infty} TZ (t; T)\, dT\,, t > 0\,, \tag{24}$$

and the initial condition

$$\langle \xi (q, 0) \nabla = \int_{-\infty}^{+\infty} T\Phi (t; T)\, dT\,, q \in [q_0, q_N]\,. \tag{25}$$

Proof. If we integrate both sides of Eq. (19) with respect to the variable T and use conditions (19) and (20) at infinity, we obtain the initial-boundary-value problem for the heat conduction equation with a zero right-hand side for the integral of the probability density function. Integration of the initial and boundary conditions leads to $\int_{-\infty}^{+\infty} \Pi_\xi dT \equiv 1$ at the boundaries and at $t = 0$. Hence, because of the homogeneity of the equation obtained for $\int_{-\infty}^{+\infty} \Pi_\xi dT$ based on the theorem of existence and uniqueness of the solution for the heat conduction equation [11], it follows that inside the region the integral is $\int_{-\infty}^{+\infty} \Pi_\xi(q, t; T)dT \equiv 1$, too.

The proof that the mean value of the thermal field obeys Eq. (22) is analogous to the proof of a similar theorem in [6]. Boundary conditions (23)-(25) are obtained from conditions (12)-(14) by multiplication of the latter by T and subsequent integration with respect to T between infinite limits.

It should be noted that Eq. (11) in the case of transformation of the Markovian field into a Markovian process, i.e., in the case of a probability density function independent of the space coordinates $\Pi_\xi(q, t; T) = \Pi_\xi(t; T)$, goes over into the well-known Kolmogorov–Fokker–Plank equation [1].

Conclusion. For the obtained equation for the probability density function it is easy to indicate a rather broad spectrum of applications in areas where the stochasticity of the flow of the described phenomenon is significant. Thus, for example, important relations can be obtained for the ignition temperature in the combustion problem of [12] and [13], in some thermal problems of nuclear power engineering [14], and when synergism methods are used in the analysis of some chemical, biological, and medical processes [15-18].

NOTATION

q_i, space node; h, increment in the space coordinate, m; t_j, time node; τ, increment in the time coordinate, sec; T, temperature, K; a, thermal diffusivity, m^2/sec; $\Pi_\xi(\bar{q}, t; T)$, probability density function; $f(q, t, T)$, source function of the thermal field (drift coefficient of the Markovian field); $B(q, t, T)$, diffusion coefficient of the Markovian field.

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